Indag. Mathem., N.S., 12 (4), 597–607 December 17, 2001

Biholomorphically invariant families amongst Carleson class

by Jie Xiao*

Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneve Blvd. West, H3G 1M8 Montreal, Quebec, Canada
e-mail: jxiao@mathstat.concordia.ca

Communicated by Prof. M.S. Keane at the meeting of September 24, 2001

ABSTRACT

Given $\alpha \in (0,1)$, let T_α be the Carleson class of all meromorphic maps f from the unit disk $\mathbb{D}$ to the extended complex plane $\bar{\mathbb{C}}$ with

$$\|f\|_{T_\alpha} = \left(\int_{\mathbb{D}} (f^{\#}(z))^2(1-|z|^2)^{1-\alpha}dm(z)\right)^{1/2} < \infty,$$

where $f^{\#}$ and dm mean the spherical derivative of f and Lebesgue area measure on $\mathbb{D}$ separately. And, let BIT_α and $BIT_{\alpha,0}$ be the biholomorphically invariant families (amongst the Carleson class) consisting of those $f \in T_\alpha$ with $\sup_{w\in\mathbb{D}} \|f\circ\phi_w\|_{T_\alpha} < \infty$ and $\lim_{|w|\to 1} \|f\circ\phi_w\|_{T_\alpha} = 0$ respectively, where $\phi_w(z) = (w-z)(1-\bar{w}z)^{-1}$, $z, w \in \mathbb{D}$. The main purpose of this article is to study BIT_α and $BIT_{\alpha,0}$ via the Ahlfors-Shimizu characteristic, canonical factorization and bounded holomorphic maps.

1. INTRODUCTION

One of fundamental problems in any area of mathematics is to determine the invariance of the structures under consideration. As for complex-functional analysis, one is interested, for instance, in studying the biholomorphically invariant families among a class consisting of some meromorphic maps. This problem is addressed here for the Carleson class.

Throughout this paper, suppose that $\mathbb{D}$, $\mathbb{T}$, $\mathbb{C}$, $\bar{\mathbb{C}}$ and dm are the unit disk, unit circle, finite complex plane, extended complex plane and two-dimensional Le-

* This research was supported by the AvH Foundation of Germany

besgue measure respectively. As usual (cf. [GrKr]), $\mathrm{Aut}(\mathbb{D})$ stands for the group of all biholomorphic maps of $\mathbb{D}$ onto itself. Any element $\phi \in \mathrm{Aut}(\mathbb{D})$ has then the representation $\zeta\phi_w$, where $\zeta \in \mathbb{T}$ and $\phi_w(z) = (w-z)(1-\bar{w}z)^{-1}$, $w, z \in \mathbb{D}$. The map ϕ_w is often used to define the pseudohyperbolic disk $\mathbb{D}(w,r) = \{z \in \mathbb{D} : |\phi_w(z)| < r\}$, $w \in \mathbb{D}$, $r \in (0,1]$. Also, assume that $\mathcal{M}$ is the class of all meromorphic maps $f : \mathbb{D} \to \bar{\mathbb{C}}$ with the spherical derivative $f^\# = |f'|(1+|f|^2)^{-1}$. Denote by $\mathcal{H}$ the class of all holomorphic maps $f : \mathbb{D} \to \mathbb{C}$, and $\mathcal{H}^\infty$ the subclass of all $f \in \mathcal{H}$ with

$$\|f\|_{\mathcal{H}^\infty} = \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

The space $\mathcal{H}^\infty$ is the typical class enjoying the biholomorphical invariance:

$$\|f \circ \phi\|_{\mathcal{H}^\infty} = \|f\|_{\mathcal{H}^\infty}, \quad \phi \in \mathrm{Aut}(\mathbb{D}).$$

Concerning $f \in \mathcal{M}$ and $r \in (0,1]$, we write $A(r,f)$ for the spherical area of the image $f(\mathbb{D}(0,r))$ with multiplicity, namely,

$$A(r,f) = \int_{\mathbb{D}(0,r)} (f^\#(z))^2 dm(z).$$

Furthermore, we say that f belongs to the Carleson class T_α, $\alpha \in [0,1]$ (cf. [Ca]) provided $\Upsilon_\alpha(f) < \infty$, where

$$\Upsilon_\alpha(f) = \begin{cases} \int_0^1 A(r,f)(1-r^2)^{-\alpha} dr, & \alpha \in [0,1); \\ \lim_{r\to 1} A(r,f), & \alpha = 1. \end{cases}$$

Integrating by parts, we find that f is of T_α class if and only if

$$\|f\|_{T_\alpha}^2 = \int_{\mathbb{D}} (f^\#(z))^2 (1-|z|^2)^{1-\alpha} dm(z) < \infty.$$

It is worth mentioning that all T_α form a family of interpolating classes between T_1 and T_0, nested by inclusion: $T_\beta \subset T_\alpha$ for $\beta > \alpha$. Nevertheless, the most important observation is that only T_1 is the class of those meromorphic maps that send $\mathbb{D}$ onto a region with finite spherical area counting multiplicity. Accordingly, when $f \in T_1$ one has the biholomorphic invariance:

$$\|f \circ \phi\|_{T_1} = \|f\|_{T_1}, \quad \phi \in \mathrm{Aut}(\mathbb{D}).$$

This leads to the following consideration. In case of $\alpha \in [0,1]$, we say that a T_α-map f belongs to BIT_α if

$$\|f\|_{BIT_\alpha} = \sup_{w\in\mathbb{D}} \|f \circ \phi_w\|_{T_\alpha} < \infty.$$

Moreover, if $\|f \circ \phi_w\|_{T_\alpha}$ tend to zero as $w \to \mathbb{T}$, then we say $f \in BIT_{\alpha,0}$. In particular, $BIT_1 = T_1$ and $BIT_{1,0} = \mathbb{C}$. It is easy to see that $T_1 \subset BIT_{\alpha,0} \subset BIT_\alpha$ for $\alpha \in [0, 1)$, and that the expected biholomorphic invariance holds true:

$$\|f \circ \phi\|_{BIT_\alpha} = \|f\|_{BIT_\alpha}, \quad \phi \in \mathrm{Aut}(\mathbb{D}).$$

Therefore, BIT_α and $BIT_{\alpha,0}$ are two biholomorphically invariant subclasses of T_α.

The famous quotient theorem of R. Nevanlinna (cf. [Ne, p. 188]) says: Every map in T_0 is the quotient of two maps in $\mathcal{H}^\infty$. In [Ca, p. 39], L. Carleson found a partial analogue of this result for T_α, and showed that each T_α-member is a quotient of two $\mathcal{H}^\infty \cap T_\beta$-maps for all $\beta < \alpha$, and conjectured that *one cannot take* $\beta = \alpha$. In 1992, A. Aleman (cf. [Al]) modified a method from the paper [RSh] of S. Richer and A. Shields to prove that every T_α-map is the quotient of two $\mathcal{H}^\infty \cap T_\alpha$-maps. In the light of this fact, the Carleson's conjecture was settled in the negative sense. Since we are interested in the biholomorphically invariant classes, we will not only seek an analogue of the Aleman's result for BIT_α resp. $BIT_{\alpha,0}$, but also explore further information about them in a way of emphasizing the biholomorphical invariance.

This paper is organized as follows. In Section 2, we characterize those maps in BIT_α resp. $BIT_{\alpha,0}$ by means of *the biholomorphically invariant form of the Ahlfors-Shimizu characteristic*. In Section 3 we give the inner-outer factorizations of these classes via *the difference between log-moduli of maps and their Poisson extensions*. Finally, in Section 4, we use the techniques developed in Sections 2 and 3 to show that each map in BIT_α resp. $BIT_{\alpha,0}$ is *the ratio of two maps* in $\mathcal{H}^\infty \cap BIT_\alpha$ resp. $\mathcal{H}^\infty \cap BIT_{\alpha,0}$.

The author is grateful to M. Essén, J.L. Fernández, F. Jafari, A. Siskakis and K.J. Wirths for some useful conversations regarding this work, as well as to the coordinating editor (M. Keane) and the referee for several suggestions that improved the readability of the paper.

2. CHARACTERIZATION

We start with two biholomorphically invariant measures. For $z \in \mathbb{D}$ and $\zeta \in \mathbb{T}$ let

$$d\lambda(z) = \frac{dm(z)}{(1-|z|^2)^2} \quad \text{resp.} \quad d\mu_z(\zeta) = \frac{1-|z|^2}{|\zeta - z|^2} \cdot \frac{|d\zeta|}{2\pi}$$

be the hyperbolic resp. harmonic measure on $\mathbb{D}$ resp. $\mathbb{T}$. It is an easy exercise to verify the biholomorphic invariance below: For $\zeta \in \mathbb{T}, w, z \in \mathbb{D}$,

$$(2.1) \qquad d\lambda(z) = d\lambda(\phi_w(z)) \quad \text{resp.} \quad d\mu_z(\zeta) = d\mu_{\phi_w(z)}(\phi_w(\zeta))$$

Next, we employ

$$T(f) = \lim_{r \to 1^-} \int_0^r A(t, f) \frac{dt}{t}$$

to represent *the Ahlfors-Shimizu characteristic* of $f \in \mathcal{M}$. With this notation, the bounded characteristic class BC (cf. [Ne]) is defined by the set of all $f \in \mathcal{M}$ obeying $T(f) = \lim_{r \to 1} T(r,f) < \infty$.

In his paper [Ya], S. Yamashita introduced UBC and UBC_0 as two biholomorphically invariant subclasses of BC. More precisely,

$$UBC = \{f \in BC : \sup_{w \in \mathbb{D}} T(f \circ \phi_w) < \infty\};$$
$$UBC_0 = \{f \in BC : \lim_{w \to \mathbb{T}} T(f \circ \phi_w) = 0\}.$$

From [Ca, p.19], [Wu, Theorem 2.2.2], [Ya, Theorem 2.2] and [AuStXi, Theorem 5.3] it turns out that as $\alpha \in [0,1)$ the following relations hold:

$$T_0 = BC; \quad BIT_\alpha \cap \mathcal{N} \subset UBC = BIT_0 \cap \mathcal{N}; \quad BIT_{\alpha,0} \subset UBC_0 = BIT_{0,0},$$

where $\mathcal{N}$ is the biholomorphically invariant class consisting of all maps $f \in \mathcal{M}$ with

$$\|f\|_{\mathcal{N}} = \sup_{z \in \mathbb{D}} (1 - |z|^2) f^{\#}(z) < \infty.$$

Upon checking the Ahlfors-Shimizu characteristic, we can obtain an area integral formula as follows:

$$T(f \circ \phi_w) = -\int_{\mathbb{D}} (f^{\#}(z))^2 \log|\phi_w(z)| dm(z), \quad f \in T_0, \quad w \in \mathbb{D}.$$

This induces a representation of BIT_α and $BIT_{\alpha,0}$.

Theorem 2.1. *Let $\alpha \in (0,1)$ and let $f \in T_\alpha$. Then for every $w \in \mathbb{D}$,*

$$\frac{\pi \|f \circ \phi_w\|^2_{T_\alpha}}{2(1-\alpha)} \le \int_{\mathbb{D}} T(f \circ \phi_z)(1 - |\phi_w(z)|^2)^{1-\alpha} d\lambda(z) \le \frac{\pi \|f \circ \phi_w\|^2_{T_\alpha}}{2\alpha(1-\alpha)}. \tag{2.2}$$

In particular,

(i) $f \in BIT_\alpha \Longleftrightarrow \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} T(f \circ \phi_z)(1 - |\phi_w(z)|^2)^{1-\alpha} d\lambda(z) < \infty$.
(ii) $f \in BIT_{\alpha,0} \Longleftrightarrow \lim_{w \to \mathbb{T}} \int_{\mathbb{D}} T(f \circ \phi_z)(1 - |\phi_w(z)|^2)^{1-\alpha} d\lambda(z) = 0$.

Proof. By Green's theorem [Ga, p.236] we get that for every $z \in \mathbb{D}$,

$$\begin{aligned}(1 - |z|^2)^{1-\alpha} &= 2\pi^{-1} \int_{\mathbb{D}} \left(\frac{\partial^2}{\partial u \partial \bar{u}} (1 - |u|^2)^{1-\alpha}\right) \log|\phi_z(u)| dm(u) \\ &= -2\pi^{-1}(1-\alpha) \int_{\mathbb{D}} (1 - (1-\alpha)|u|^2)(1 - |u|^2)^{1-\alpha} \log|\phi_z(u)| d\lambda(u).\end{aligned}$$

Further, from the biholomorphic invariance of $d\lambda$, Fubini's theorem and the change of variables: $u = \phi_w(v)$ and $y = \phi_w(z)$ it follows that

$$
\begin{aligned}
\frac{\pi\|f\|^2_{T_\alpha}}{2(1-\alpha)} &= \int_{\mathbb{D}} (f^{\#}(y))^2\left(\int_{\mathbb{D}} \frac{((1-\alpha)|u|^2-1)\log|\phi_y(u)|}{(1-|u|^2)^{\alpha-1}}d\lambda(u)\right)dm(y)\\
&= \int_{\mathbb{D}} \frac{(1-\alpha)|\phi_w(v)|^2-1}{(1-|\phi_w(v)|^2)^{\alpha-1}}\left(\int_{\mathbb{D}} (f^{\#}(y))^2\log|\phi_y(\phi_w(v))|dm(y)\right)d\lambda(v)\\
&= \int_{\mathbb{D}} \frac{(1-\alpha)|\phi_w(v)|^2-1}{(1-|\phi_w(v)|^2)^{\alpha-1}}\left(\int_{\mathbb{D}} (f^{\#}(\phi_w(z)))^2\log|\phi_v(z)|dm(\phi_w(z))\right)d\lambda(v)\\
&= \int_{\mathbb{D}} (1-|\phi_w(v)|^2)^{1-\alpha}(1-(1-\alpha)|\phi_w(v)|^2)T(f\circ\phi_w\circ\phi_v)d\lambda(v).
\end{aligned}
$$

Replacing f with $f\circ\phi_w$ and making a simple calculation, we obtain (2.2) which implies (i) and (ii) right away. □

Remark. In fact, (2.2) reveals that $f\in T_\alpha \Longleftrightarrow \int_{\mathbb{D}} T(f\circ\phi_z)(1-|z|^2)^{1-\alpha}d\lambda(z)<\infty$, where $\alpha\in(0,1)$. Note that $\|f\|^2_{T_1} = 2\limsup_{\alpha\to 1^-}(1-\alpha)\|f\|^2_{T_\alpha}$ whenever $f\in T_1$. So, (2.2) also implies that

$$
f\in T_1 \Longleftrightarrow \limsup_{\alpha\to 1^-}(1-\alpha)^2\int_{\mathbb{D}} T(f\circ\phi_z)(1-|z|^2)^{1-\alpha}d\lambda(z)<\infty.
$$

3. FACTORIZATION

A bounded holomorphic self-map I of $\mathbb{D}$ is called *inner* provided $\|I\|_\infty\le 1$ and its radial limits $\lim_{r\to 1}|I(r\zeta)|=1$ at almost all points $\zeta\in\mathbb{T}$. An *outer* map is the map of the form:

$$
O_\psi(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\log\psi(\zeta)\frac{|d\zeta|}{2\pi}\right),
$$

where $\psi>0$ a.e. on $\mathbb{T}$ and $\log\psi\in L^1(\mathbb{T})$.

To avoid trivialities we, from now on, assume that f is a nonconstant map in $\mathcal{M}$. If $f\in T_0$ is a nonzero map, then f has a nontangential limit function, still denoted by f, a.e. on $\mathbb{T}$, but also $\log|f|$ belongs to $L^1(\mathbb{T})$. Moreover, every T_0-map f may be decomposed into the form $f=IO/J$, where I,J are inner maps whose greatest common divisor is 1 and where O is an outer map in T_0. Conversely, such a map IO/J belongs to T_0. Up to some unimodular constants, I,J are uniquely determined in this case. No matter whether $z\in\mathbb{D}$ is a pole of f or not, the following quantity is well-defined:

$$
(3.1)\qquad E(f,z) = \int_{\mathbb{T}} \log(1+|O(\zeta)|^2)d\mu_z(\zeta) - \log(|J(z)|^2+|I(z)O(z)|^2).
$$

As f is holomorphic, $-E(f,z)$ coincides with $\log(1+|f(z)|^2)$ (subharmonic on $\mathbb{D}$) minus its Poisson integral (i.e., least harmonic majorant). This quantity has indeed uncovered a geometric behavior of maps in UBC and UBC_0 (cf. [Ya, Section 5]). More is true:

Lemma 3.1. *Let $f = IO/J \in T_0$, where I, J are inner maps whose greatest common divisor is* 1*; and O is outer. Then for every $w \in \mathbb{D}$,*

$$E(f, w) = 2\pi^{-1} T(f \circ \phi_w). \tag{3.2}$$

In particular,

(i) $f \in UBC \Longleftrightarrow \sup_{w \in \mathbb{D}} E(f, w) < \infty$.

(ii) $f \in UBC_0 \Longleftrightarrow \lim_{w \to \mathbb{T}} E(f, w) = 0$.

Proof. If $w \in \mathbb{D}$ is not a pole of f then (3.2) follows from (3.1) and [Al, Lemma 2]. If, otherwise, $w \in \mathbb{D}$ is a pole of f, then one considers $1/f$ which sends w to 0. Hence (3.2) is valid for $1/f$. Since $(1/f)^{\#} = f^{\#}$ and $\log|O|$ is harmonic, one has

$$\begin{aligned} 2\pi^{-1} T(f \circ \phi_w) &= 2\pi^{-1} T\big(1/f \circ \phi_w\big) \\ &= E(1/f, w) \\ &= \int_{\mathbb{T}} \log(1 + |1/O(\zeta)|^2) d\mu_w(\zeta) - \log(|I(w)|^2 + |J(w)/O(w)|^2) \\ &= E(f, w) - 2\left(\int_{\mathbb{T}} \log|O(\zeta)| d\mu_w(\zeta) - \log|O(w)| \right) \\ &= E(f, w). \end{aligned}$$

This completes the proof. □

Remark. Under the assumption of Lemma 3.1, we have

$$E(f, w) = E(O, w) + F(I, J, O, w), \tag{3.3}$$

here and afterwards,

$$F(I, J, O, w) = \log(1 + |O(w)|^2) - \log(|J(w)|^2 + |I(w)O(w)|^2).$$

As a consequence, we get the following proposition:

$$f \in UBC \Longleftrightarrow O \in UBC \quad \text{and} \quad \sup_{w \in \mathbb{D}} F(I, J, O, w) < \infty;$$

$$f \in UBC_0 \Longleftrightarrow O \in UBC_0 \quad \text{and} \quad \lim_{w \to \mathbb{T}} F(I, J, O, w) = 0.$$

This is a formulation of the canonical factorization of UBC and UBC_0, covering [Ya, Theorem 4.1].

Theorem 3.2. *Let $\alpha \in (0, 1)$ and let $f = IO/J \in T_\alpha$, where I, J are inner maps whose greatest common divisor is* 1*; and O is outer. Define*

$$G_\alpha(I, J, O, w) = \int_{\mathbb{D}} F(I, J, O, z)(1 - |\phi_w(z)|^2)^{1-\alpha} d\lambda(z), \quad w \in \mathbb{D}.$$

Then the following are true:

(i) $f \in BIT_\alpha \Longleftrightarrow O \in BIT_\alpha$ *and* $\sup_{w\in\mathbb{D}} G_\alpha(I,J,O,w) < \infty$.
(ii) $f \in BIT_{\alpha,0} \Longleftrightarrow O \in BIT_{\alpha,0}$ *and* $\lim_{w\to\mathbb{T}} G_\alpha(I,J,O,w) = 0$.

Proof. It follows from (3.2), (3.3) and Theorem 2.1. $\square$

Remark. Theorem 2.1, together with (3.2) and (3.3), also suggests the canonical factorization of T_α, $\alpha \in (0,1)$: $f = IO/J \in T_\alpha \Longleftrightarrow O \in T_\alpha$ and $G_\alpha(I,J,O,0) < \infty$. In case of $\alpha = 1$, we use the same reasoning (as well as the remark attached to Theorem 2.1) to produce that $f = IO/J \in T_1 \Longleftrightarrow O \in T_1$ and $\limsup_{\alpha\to 1^-} G_\alpha(I,J,O,0) < \infty$.

Via analyzing the preceding discussion, we have realized it necessary to describe each inner or outer map from the different subclasses individually. It is already known that every inner map lies in T_0 (of course, UBC). However, from [St, Section 3] it turns out that an inner map is in UBC_0 if and only if it is a finite Blaschke product. When $\alpha \in (0,1)$, P. R. Ahern's article [Ah] shows that an inner map I lies in T_α if and only if there exists a set $A \subset \mathbb{D}$ of logarithmic capacity zero such that

$$\sum_{I(z)=w} (1-|z|^2)^{1-\alpha} < \infty, \quad w \in \mathbb{D}\setminus A.$$

In contrast with this fact, Theorems 1.2 and 5.6 in [EX] tell us that an inner map I belongs to BIT_α or $BIT_{\alpha,0}$ if and only if I is a Blaschke product satisfying

$$\sup_{w\in\mathbb{D}} \sum_{I\circ\phi_w(z)=0} (1-|z|^2)^{1-\alpha} < \infty \quad \text{or} \quad \mathrm{Card}(\{z : I(z)=0\}) < \infty.$$

To characterize the outer maps in BIT_α and $BIT_{\alpha,0}$, we make some additional notations below: For $\psi > 0$ a.e. on $\mathbb{T}$, $z \in \mathbb{D}$ and $\log\psi \in L^1(\mathbb{T})$, put

$$\Psi(z) = \exp\Big(\int_{\mathbb{T}} \log\psi(\zeta)d\mu_z(\zeta)\Big)$$

and

$$S(\psi,z) = \int_{\mathbb{T}} \log(1+|\psi(\zeta)|^2)d\mu_z(\zeta) - \log\big(1+(\Psi(z))^2\big).$$

It is evident that $|O_\psi(z)| = \Psi(z)$ for $z \in \mathbb{D}$ and $|O_\psi(\zeta)| = \psi(\zeta)$ for a.e $\zeta \in \mathbb{T}$. Thus, an application of (3.2) infers that $O_\psi \in T_0 \Longleftrightarrow S(\psi,0) < \infty$, and so that

$$O_\psi \in UBC \Longleftrightarrow \sup_{z\in\mathbb{D}} S(\psi,z) < \infty; \quad O_\psi \in UBC_0 \Longleftrightarrow \lim_{z\to\mathbb{T}} S(\psi,z) = 0.$$

Theorem 3.3. *Let* $\alpha \in (0,1)$ *and let* $\psi > 0$ *a.e. on* $\mathbb{T}$ *and* $\log\psi \in L^1(\mathbb{T})$. *Define*

$$R_\alpha(\psi,w) = \int_{\mathbb{D}} S(\psi,z)(1-|\phi_w(z)|^2)^{1-\alpha}d\lambda(z), \quad w \in \mathbb{D}.$$

Then the following are true:

(i) $O_\psi \in BIT_\alpha \Longleftrightarrow \sup_{w\in\mathbb{D}} R_\alpha(\psi, w) < \infty$.

(ii) $O_\psi \in BIT_{\alpha,0} \Longleftrightarrow \lim_{w\to\mathbb{T}} R_\alpha(\psi, w) = 0$.

Proof. Observing once again that $|O_\psi| = \Psi$ on $\mathbb{D}$ and $|O_\psi| = \psi$ a.e on $\mathbb{T}$, we apply (3.2) to deduce that for any $z \in \mathbb{D}$,

$$\begin{aligned} 2\pi^{-1}T(O_\psi \circ \phi_z) &= \int_{\mathbb{T}} \log(1 + |O_\psi(\zeta)|^2)d\mu_z(\zeta) - \log(1 + |O_\psi(z)|^2) \\ &= \int_{\mathbb{T}} \log(1 + |\psi(\zeta)|^2)d\mu_z(\zeta) - \log(1 + (\Psi(z))^2) \\ &= S(\psi, z), \end{aligned}$$

which, together with Theorem 2.1, implies (i) and (ii), as desired. □

Remark. The above argument yields that $O_\psi \in T_\alpha \Longleftrightarrow R_\alpha(\psi, 0) < \infty$ for $\alpha \in (0, 1)$, and that $O_\psi \in T_1 \Longleftrightarrow \limsup_{\alpha\to 1^-}(1 - \alpha)^2 R_\alpha(\psi, 0) < \infty$.

4. QUOTIENT

We begin by citing a result of A. Aleman.

Lemma 4.1. *Let (X, μ) be a probability space and let $\psi \in L^1(\mu)$ with $\psi > 0$ μ-a.e. on X and $\log\psi \in L^1(\mu)$. Define*

$$E_\gamma(\psi) = \int_X \log(1 + \psi)d\mu - \log\left(1 + \gamma \exp \int_X \log\psi d\mu\right), \quad \gamma \in [0, 1].$$

Then

$$E_\gamma(\min\{1, \psi\}) \leq E_\gamma(\psi).$$

Proof. See Lemma 4 in [Al].

In the sequel, an outer map O is associated with two cut-off outer maps:

$$O_+(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log(\max\{1, |O(\zeta)|\}) \frac{|d\zeta|}{2\pi}\right)$$

and

$$O_-(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log(\min\{1, |O(\zeta)|\}) \frac{|d\zeta|}{2\pi}\right).$$

It is clear that both O_- and $1/O_+$ lie in $\mathcal{H}^\infty$. However, a key observation is:

$$O(z) = O_+(z)O_-(z), \quad |O_-(z)| \leq |O(z)|, \quad |O_+(z)| \geq 1, \quad z \in \mathbb{D}.$$

Lemma 4.2. *Let $f = IO/J \in T_0$, where I, J are inner maps whose greatest com-*

mon divisor is 1*; and O is outer. Also let Y be either UBC or UBC_0. If f is in Y then so are IO_- and J/O_+ with*

$$(4.1)\qquad E(IO_-,z)\le E(f,z);\quad E(J/O_+,z)\le E(f,z),\quad z\in\mathbb{D}.$$

Hence every Y-map can be written as the quotient of two $\mathcal{H}^\infty\cap Y$-maps.

Proof. Since $|(IO_-)(\zeta)|=\min\{1,|O(\zeta)|\}$ at a.e. $\zeta\in\mathbb{T}$, an application of (3.1), as well as Lemma 4.1 with $X=\mathbb{T}$, $d\mu=d\mu_z$, $\gamma=|I(z)|^2$ and $\psi=\min\{1,|O(\zeta)|\}$, gives that for any fixed $z\in\mathbb{D}$,

$$\begin{aligned}
E(IO_-,z)&=\int_{\mathbb{T}}\log(1+|I(\zeta)O_-(\zeta)|)d\mu_z(\zeta)-\log(1+|I(z)O_-(z)|^2)\\
&=E_{|I(z)|^2}(\min\{1,|O|^2\})\\
&\le E_{|I(z)|^2}(|O|^2)\\
&\le\int_{\mathbb{T}}\log(1+|O(\zeta)|^2)d\mu_z(\zeta)-\log(|J(z)|^2+|I(z)O(z)|^2)\\
&=E(f,z).
\end{aligned}$$

Meantime, we also have

$$\begin{aligned}
E(J/O_+,z)&=\int_{\mathbb{T}}\log(1+|J(\zeta)/O_+(\zeta)|)d\mu_z(\zeta)-\log(1+|J(z)/O_+(z)|^2)\\
&=E_{|J(z)|^2}(\min\{1,1/|O|^2\})\\
&\le E_{|J(z)|^2}(1/|O|^2)\\
&\le\int_{\mathbb{T}}\log(1+|1/O(\zeta)|^2)d\mu_z(\zeta)-\log(|I(z)|^2+|J(z)/O(z)|^2)\\
&=E(1/f,z)\\
&=E(f,z).
\end{aligned}$$

Thus (4.1) follows. Thanks to the equation $f=IO_-/(J/O_+)$, every map in Y is the quotient of two maps in $\mathcal{H}^\infty\cap Y$. The proof is complete. □

The last proof provides actually a new approach to demonstrate the Nevanlinna's quotient theorem. At this point, it is very natural for us to establish the following quotient representation theorem for BIT_α and $BIT_{\alpha,0}$.

Theorem 4.3. *Let $\alpha\in(0,1)$ and let $f=IO/J\in T_\alpha$, where I,J are inner maps whose greatest common divisor is* 1*; and O is outer. If f belongs to BIT_α resp. $BIT_{\alpha,0}$ then both IO_- and J/O_+ lie in BIT_α resp. $BIT_{\alpha,0}$ with*

$$(4.2)\qquad \|IO_-\|_{BIT_\alpha}\le\alpha^{-1/2}\|f\|_{BIT_\alpha};\quad \|J/O_+\|_{BIT_\alpha}\le\alpha^{-1/2}\|f\|_{BIT_\alpha}.$$

Consequently, every map in BIT_α resp. $BIT_{\alpha,0}$ can be factored as the quotient of two maps in $\mathcal{H}^\infty\cap BIT_\alpha$ resp. $\mathcal{H}^\infty\cap BIT_{\alpha,0}$.

Proof. We need only verify (4.2). By using (2.2), (3.2) twice and (4.1) once, we have

$$
\begin{aligned}
\|IO_-\|^2_{BIT_\alpha} &\leq 2\pi^{-1}(1-\alpha)\sup_{w\in\mathbb{D}}\int_{\mathbb{D}} T\Big((IO_-)\circ\phi_z\Big)(1-|\phi_w(z)|^2)^{1-\alpha}d\lambda(z)\\
&= (1-\alpha)\sup_{w\in\mathbb{D}}\int_{\mathbb{D}} E(IO_-,z)(1-|\phi_w(z)|^2)^{1-\alpha}d\lambda(z)\\
&\leq (1-\alpha)\sup_{w\in\mathbb{D}}\int_{\mathbb{D}} E(f,z)(1-|\phi_w(z)|^2)^{1-\alpha}d\lambda(z)\\
&= 2\pi^{-1}(1-\alpha)\sup_{w\in\mathbb{D}}\int_{\mathbb{D}} T(f\circ\phi_z)(1-|\phi_w(z)|^2)^{1-\alpha}d\lambda(z)\\
&\leq \alpha^{-1}\sup_{w\in\mathbb{D}}\|f\circ\phi_w\|^2_{T_\alpha}\\
&= \alpha^{-1}\|f\|^2_{BIT_\alpha}.
\end{aligned}
$$

In a similar manner, we also obtain the following size estimates:

$$
\begin{aligned}
\|J/O_+\|^2_{BIT_\alpha} &\leq 2\pi^{-1}(1-\alpha)\sup_{w\in\mathbb{D}}\int_{\mathbb{D}} T\Big((J/O_+)\circ\phi_z\Big)(1-|\phi_w(z)|^2)^{1-\alpha}d\lambda(z)\\
&= (1-\alpha)\sup_{w\in\mathbb{D}}\int_{\mathbb{D}} E(J/O_+,z)(1-|\phi_w(z)|^2)^{1-\alpha}d\lambda(z)\\
&\leq (1-\alpha)\sup_{w\in\mathbb{D}}\int_{\mathbb{D}} E(f,z)(1-|\phi_w(z)|^2)^{1-\alpha}d\lambda(z)\\
&= 2\pi^{-1}(1-\alpha)\sup_{w\in\mathbb{D}}\int_{\mathbb{D}} T(f\circ\phi_z)(1-|\phi_w(z)|^2)^{1-\alpha}d\lambda(z)\\
&\leq \alpha^{-1}\sup_{w\in\mathbb{D}}\|f\circ\phi_w\|^2_{T_\alpha}\\
&= \alpha^{-1}\|f\|^2_{BIT_\alpha}.
\end{aligned}
$$

Therefore, the proof is complete. □

Remark. There is, of course, a variant of Theorem 4.3 valid in T_α, $\alpha\in(0,1]$; see also [Al, Theorem 1].

REFERENCES

[Ah] Ahern, P.R. – The mean modulus and the derivative of an inner function. Indiana Univ. Math. J. **28**, 311–347 (1979).

[Al] Aleman, A. – A conjecture of L. Carleson and applications. Michigan Math. J. **39**, 537–549 (1992).

[AuStXi] Aulaskari, R. D. Stegenga and J. Xiao – Some subclasses of BMOA and their characterization in terms of Carleson measures. Rocky Mountain J. Math. **26**, 485–506 (1996).

[Ca] Carleson, L. – On a class of meromorphic functions and its associated exceptional sets, Thesis, University of Uppsala (1950).

[EX] Essén, M. and J. Xiao – Some results on Q_p spaces, $0<p<1$. J. Reine Angew. Math. **485**, 173–195 (1997).

[Ga] Garnett, J. – Bounded Analytic Functions. Academic Press (1981).

[GrKr] Greene, R.E. and S. G. Krantz – Biholomorphic self-maps of domains. Lecture Note in Math. **1276**, 136–207.

[Ne] Nevanlinna, R. – Analytic Functions. Springer-Verlag, Berlin-Heidelberg, New York (1970).

[RSh] Richter, S. and A.L Shields – Bounded analytic functions in Dirichlet space. Math. Z. **198**, 151–159 (1998).

[St] Stegenga, D.A. – Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation. Amer. J. Math. **98**, 573–589 (1976).

[Wu] Wulan, H. – On some classes of meromorphic functions. Ann. Acad. Sci. Fenn. Math. Diss., **116** (1998).

[Ya] Yamashita, S. – Functions of uniformly bounded characteristic. Ann. Acad. Sci. Fenn. Ser. A.I. Math. **7**, 349–367 (1982).

(Received January 2000)